The Algorithmics of Solitaire-Like Games

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Abstract. Puzzles and games have been used for centuries to nurture problem-solving skills. Although often presented as isolated brainteasers, the desire to know how to win makes games ideal examples for teaching algorithmic problem solving. With this in mind, this paper explores one-person solitaire-like games.
The key to understanding solutions to solitaire-like games is the identification of invariant properties of polynomial arithmetic. We demonstrate this via three case studies: solitaire itself, tiling problems and a collection of novel one-person games. The known classification of states of the game of (peg) solitaire into 16 equivalence classes is used to introduce the relevance of polynomial arithmetic. Then we give a novel algebraic formulation of the solution to a class of tiling problems. Finally, we introduce an infinite class of challenging one-person games inspired by earlier work by Chen and Backhouse on the relation between cyclotomic polynomials and generalisations of the seven-trees-in-one type isomorphism. We show how to derive algorithms to solve these games.

Keywords: Solitaire, invariants, tiling problems, polynomials, games on cyclotomic polynomials, seven-trees-in-one, nuclear pennies, algorithm derivation

1 Introduction

There are two concepts that are basic to all algorithms that process input values using some sort of iterative scheme: invariants and making progress. Although in principle making progress can involve quite complicated theories on well-founded relations, in practice the concept is easy for students to grasp. On the other hand, students are often given misleading information about invariants; they are often taught that invariants are (only) needed for *post-hoc* verification of program correctness and very difficult to formulate. In reality, a good understanding of invariants is crucial to successful algorithm design.

For the last seven years, the module Algorithmic Problem Solving has been taught to first-year Computer Science students at the University of Nottingham. The module aims to introduce students to effective problem-solving techniques,

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particularly in the context of solving problems that demand an algorithmic solution; the first technique that is presented is the use of invariants. At a later stage, two-person games are studied in some depth; games are inherently algorithmic in nature (after all, the goal is to win, which means formulating some sort of algorithm) and require little motivation.

There are many puzzles and games in the mathematical literature which have been used for centuries to nurture problem-solving ability. Many are presented as isolated brain-teasers but, for our pedagogical purposes, it is important that they have two qualities. First, any problem that is studied must have a substantial number of variations which can be used to test students' understanding and, second, the solution method should demonstrate effective algorithmic problem-solving rather than being ad hoc or magic.

Recently, we have been studying one-person solitaire-like games in order to try to extract useful examples for study in the module. In this paper, we present our findings so far, including an infinite class of challenging games that we have invented based on insights from type theory.

We begin the paper in section 2 with a brief summary of well-known properties of the game of solitaire. These properties are derived using the algebra of polynomials in a suitably chosen semiring; it is this algebra that is the basis for the novel applications that we discuss in later sections.

Section 3 is about a class of tiling problems. In section 3.1, we show how Golomb's [1] use of colours to solve one such problem is formulated algebraically. (Our solution is simpler than the algebraic formulation proposed by Mackinnon [2].) The solution to the class of tiling problems is discussed in section 3.2.

The so-called "nuclear pennies" game [3] is an example of a game which, until now, has been of isolated interest. The game is based on the theorem attributed to Lawvere that "seven trees are one". That is, if T is the type of complete binary trees, the type T^7 (the Cartesian product of T with itself 7 times) is isomorphic to T. The game involves moving a checker 6 places to the right on a one-dimensional tape (from position 1 to position 7) following rules that reflect the recursive definition of binary trees. In section 4, we formulate an infinite collection of games, each with different rules, where the goal is to move a checker a certain number of positions from its starting position on a one-dimensional tape. So far as we are aware, these games and their solution are original to this paper. The games were derived from our study of the problem: given a number n, invent an interesting type T such that T^n is isomorphic to T [4].

2 Solitaire and Variations

Solitaire is a well-known game. The game begins with a number of pegs stuck in holes in a board. The holes are arranged in a grid, the shape of which is not relevant to the current discussion. A move, shown diagrammatically in fig. 1, replaces two pegs by one and the game is to remove all pegs bar one, leaving the peg in a designated position. In this section, we show how invariants of polynomial arithmetic are used in the analysis of moves in the game of solitaire and in a variation on solitaire called the solitaire army game.

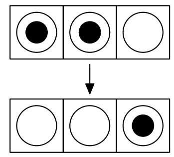


Fig. 1. Move from left to right. Similar moves are allowed from right to left, from top to bottom, and from bottom to top.

2.1 Solitaire

De Bruijn [5] shows that states in the game of solitaire can be divided into 16 equivalence classes in such a way that all moves are between equivalent states. (So the equivalence class of the state is an invariant of each move.) Here is a brief reformulation of De Bruijn's argument[1].

Suppose we assign non-negative integer coordinates (i,j) to each hole in the board. Suppose $R=(A,\mathbf{0},\mathbf{1},+,\cdot)$ is a semiring[2] and suppose p is an element of A. Assign to a peg at position (i,j) the *weight* p^{i+j}. The *total weight* of a state in the game is the sum of the weights of all the pegs on the board in that state. There are four types of move in the game — vertically up and down, and horizontally left and right. A vertical-up move replaces pegs with weights p^{i+j+0} and p^{i+j+1} by a peg with weight p^{i+j+2}. So, if p has the property that $p^0+p^1=p^2$, the total weight is invariant. Similarly, a horizontal-left move replaces pegs with weights p^{i+2+j} and p^{i+1+j} by a peg with weight p^{i+0+j}. So, if $p^2+p^1=p^0$, the total weight remains invariant. A similar analysis applies to the two other types of moves: if $p^2+p^1=p^0$, the total weight is invariant under vertical-down moves and, if $p^0+p^1=p^2$, the total weight is invariant under horizontal-right moves.

The carrier set of the field[3] $GF(4)$ has exactly 4 elements which can be named $\mathbf{0}$, $\mathbf{1}$, p and p^2. Moreover, these elements have the property that $\mathbf{1}+\mathbf{1}=\mathbf{0}$, $\mathbf{1}+p=p^2$ and (hence) $p^2+p=\mathbf{1}$. Thus, if $GF(4)$ is used to compute the weights of states, the weight is invariant under all moves. This divides the states into four equivalence classes, and the initial and final states in the game must be in the same equivalence class.

[1] Berlekamp *et al* [6, pp. 708–710] attribute the theorem to M. Reiss [7] but the argument is different. De Bruijn's argument is more relevant to later sections of this paper. We have not read Reiss's paper but assume that Berlekamp *et al* copy his presentation.

[2] That is, addition in R is associative and commutative and has unit $\mathbf{0}$, product is associative and has unit $\mathbf{1}$ and zero $\mathbf{0}$, and product distributes over addition.

[3] The fact that $GF(4)$ is a field and not just a semiring is not relevant to the argument.

To complete the argument, a symmetrical weighting is used: assign to a peg at position (i,j) the weight p^{i-j}. The total weight is again the sum of the weights of all the pegs on the board in that state. (We call it a "symmetrical" weighting because it is equivalent to turning the board through 90^0.) The same analysis applies, classifying each state into one of 4 equivalence classes.

Combining the two[4], the states are divided into $4{\times}4$ equivalence classes in such a way that the equivalence class is invariant under moves. The game can be solved only if the initial and final states are in the same equivalence class.

2.2 The Solitaire Army

De Bruijn's weighting of a state in a game does not provide a sufficient condition for when it is possible to move from a given initial state to a given final state. Berlekamp *et al* [6, chap. 23] discuss in detail a number of problems, when they can be solved and when they cannot be solved. A tool in their analysis is the notion of a *pagoda function*, which computing scientists would recognise as being similar to a measure of progress. Specifically, a pagoda function is any real-valued function pag on peg-positions that has the property that if a move replaces pegs at positions r and s by a peg at position t then

$$pag.t \leq pag.s + pag.r$$

A much-celebrated problem —discussed in several other books and Internet pages— which they solve using a pagoda function is the "solitaire army" problem. This is how they describe the problem.

> A number of Solitaire men stand initially on one side of a straight line beyond which is an infinite empty desert. How many men do we need to send a scout just 0, 1, 2, 3, 4 or 5 paces out into the desert?

The surprising fact is that for 5 paces (or more) there is no solution! The proof (attributed in Wikipedia to John Horton Conway, 1961) resembles De Bruijn's analysis. Suppose peg positions are assigned Cartesian coordinates so that the goal position is given the coordinates $(0,0)$ and the initial positions of the Solitaire men have negative coordinates. Suppose we assign to a peg at position (i,j) the weight $\sigma^{|i+j|}$, where σ is yet to be chosen. Note that $|i+j|$ is the distance of (i,j) from $(0,0)$ along a shortest path taken by a peg in moves of the game. The weight of any state in the game is the sum of the weights of all the pegs on the board in that state. Now σ is chosen so that the weighting of pegs is a pagoda function. Specifically, choose $\sigma = \frac{1}{2}(\sqrt{5}-1)$ so that $\sigma^2+\sigma = 1$. (This guarantees that the weighting of pegs remains constant when a peg is moved along a shortest path to $(0,0)$ and decreases for other moves.) Then the maximum weight of an initial state in which a finite number

[4] As pointed out to us by Diethard Michaelis, the combination of weights is an element of the semiring $GF(4){\times}GF(4)$ where addition and product are defined componentwise. Note that $GF(4){\times}GF(4)$ is not a field because it has divisors of zero.

of Solitaire men is at a distance at least n from $(0,0)$ is less than σ^{n-5} . For the 5 -pace problem, the goal is to reach a state with weight at least σ^{5-5} (i.e. 1) but this is impossible because the weighting is a pagoda function — its value is never increased by a move.

Edsger W. Dijsktra [8] discusses a similar problem (and gives a similar solution).

3 Tiling Problems

Tiling problems involve covering a board without overlapping with a given collection of tiles. Traditionally their solution involves (seemingly ad hoc) colouring arguments. This section is essentially about how to formulate the colouring arguments algebraically.

3.1 The Chessboard Problem

Consider the problem of tiling a chessboard with twenty-one 3×1 rectangles and one 1×1 square. Index each square of the chessboard by a pair of natural numbers (i,j) in the obvious way. For concreteness, we assume that the bottom-left corner is given the label $(0,0)$.

Suppose a chessboard is partially tiled by 3×1 rectangles. As in De Bruijn's analysis of solitaire, give to the square (i,j) two "weights": the *forward* weight is p^{i-j} and the *backward* weight is p^{i+j} , where p is a generator of the field $GF(4)$. Two weights are assigned to the chessboard as follows: the *forward weight* of the chessboard is the sum (in $GF(4)$) of the forward weights of all the individual squares that are tiled, and the *backward weight* of the chessboard is the sum of the backward weights of all the individual squares that are tiled.

Recall that the elements of $GF(4)$ are $\mathbf{0}$, $\mathbf{1}$, p and p^2 and that $\mathbf{1}+\mathbf{1}=\mathbf{0}$ and $\mathbf{1}+p=p^2$. It follows that

(1) $\qquad \mathbf{0}=\mathbf{1}+p+p^2$.

In particular, $p^3=\mathbf{1}$, the forward weight of square (i,j) is $p^{(i-j)\ \mathsf{mod}\ 3}$ and its backward weight is $p^{(i+j)\ \mathsf{mod}\ 3}$. Thus the weights are identical on forward and backward diagonals of the board, respectively. (This is the explanation for our choice of nomenclature.)

It is easily checked that when a 3×1 tile is placed on a chessboard, both the forward and backward weights of the chessboard do not change; they are *invariants* of the tiling process. (For example, if a 3×1 tile is placed horizontally on the board with leftmost square at position (i,j) , the weight $p^{i+j}\times(\mathbf{1}+p+p^2)$ is added to the weight of the board. Because $\mathbf{1}+p+p^2$ equals $\mathbf{0}$, adding or subtracting this weight has no effect on the total weight.) This is the basis for the choice of $GF(4)$ in weighing squares: it is the simplest possible semiring that satisfies (1) in a non-trivial way.

The forward and backward weights of a completely tiled chessboard are 1 and p , respectively. In order to tile the chessboard completely with twenty-one 3×1 rectangles and one 1×1 square, the 1×1 square must therefore be placed on a square with forward weight 1 and backward weight p. The former are the squares (i,j) with $(i-j\equiv 0)\ \mathsf{mod}\ 3$ and the latter are the squares (i,j) with $(i+j\equiv 1)\ \mathsf{mod}\ 3$. Combined with the requirement that i and j are natural numbers each of which is at most 7, there are just 4 solutions to this pair of equations, namely $(i,j)=(2,2)$, $(i,j)=(5,5)$, $(i,j)=(2,5)$, and $(i,j)=(5,2)$.

The above argument is a simpler presentation of an "algebraic" proof given by Mackinnon [2]. (Our formulation is simpler because Mackinnon takes for p a complex solution of equation (1) — the field of complex numbers is, of course, much more complicated than $GF(4)$.) If colours —say red, white and blue— are assigned to the non-zero elements of $GF(4)$, the argument is essentially the same as Golomb's [1] "colouring" proof. Specifically, Golomb's proof has two components, the colouring of the squares and rotational symmetry of the board. The colouring of the squares is just the assignment of three different values to the squares; this is chosen so that the net "count" of colours on the board —whereby three differently coloured squares "count" as zero— is one. The rotational symmetry is expressed algebraically by the two weights given to squares of the board. The colouring and algebraic proofs are thus in essence identical.

3.2 The Generalisation

In order to demonstrate the effectiveness of the algebraic formulation, let us consider a generalisation. Suppose we have an $m\times m$ board, an unlimited supply of n-ominoes and one 1-omino. (An n-omino is an $n\times 1$ board, i.e. a strip of n squares each of which is the same size as a square of the given $m\times m$ board.) In order to eliminate the trivial case, we assume that $1<m$. We prove that it is possible to cover the $m\times m$ board with the supplied n-ominoes and one 1-omino, without overlapping, precisely when

(2) $\qquad 1\leq n<m\ \wedge\ (n\backslash(m-1)\vee n\backslash(m+1))\quad .$

An obvious necessary condition is

$$1\leq n<m\ \wedge\ n\backslash(m^2-1)\quad .$$

This, however, is not equivalent. For example, it is satisfied by $m=11$ and $n=8$ but it is not the case that $8\backslash(11-1)\vee 8\backslash(11+1)$.

Invariant. Arbitrary Semiring First, we show that (2) is necessary. Consider any semiring $R=(A,\mathbf{0},\mathbf{1},+,\cdot)$ with an element $x\in A$ that has the property that

(3) $\qquad \langle \Sigma i : 0\leq i<n : x^i\rangle\ =_R\ \mathbf{0}\quad .$

(We give examples of such semirings later. The subscript on the equality symbol is necessary later to avoid the confusion that can be caused by overloading.) Now let us assign to each square (i,j) the *weight* x^{i+j} if it is covered and the weight 0 if it is not covered. The weight of a (partially) tiled board is defined to be the sum of the weights of the tiled squares.

On account of (3) above, the placement of an n-omino on the board does not change the weight of the board. Since there is exactly one 1-omino on a completely covered board, a necessary condition is that

$$\langle \exists k :: \langle \Sigma i,j : 0 \le i < m \wedge 0 \le j < m : x^{i+j} \rangle =_R x^k \rangle \quad .$$

Equivalently (calculation left to the reader),

$$(4) \qquad \left\langle \exists k :: \langle \Sigma i : 0 \le i < m : x^i \rangle^2 =_R x^k \right\rangle \quad .$$

We show that (4) implies $n \backslash (m-1) \vee n \backslash (m+1)$. Our calculations exploit the following immediate consequences of (3): for all j,

$$(5) \qquad x^j =_R x^{j \,\mathsf{mod}\, n} \quad ,$$

and, hence, for all j,

$$(6) \qquad \langle \Sigma i : 0 \le i < j : x^i \rangle =_R \langle \Sigma i : 0 \le i < j \,\mathsf{mod}\, n : x^i \rangle \quad .$$

Invariant. Polynomials over $GF(2)$ To complete our argument, we fix the semiring R to be

$$GF(2)[x] \;/\; \langle \Sigma i : 0 \le i < n : x^i \rangle$$

That is, R is the set of polynomials in the indeterminate x with coefficients in $GF(2)$ (which is conventionally denoted by $GF(2)[x]$) modulo the polynomial $\langle \Sigma i : 0 \le i < n : x^i \rangle$. Thus, in R we have the property (3).

This choice of R is motivated by our goal. Note first the squaring in (4); $GF(2)$ is the simplest example of a semiring in which squaring distributes through addition. This property is easily seen to be inherited by $GF(2)[x]$. That is, for all j,

$$(7) \qquad \langle \Sigma i : 0 \le i < j : x^i \rangle^2 =_{GF(2)[x]} \langle \Sigma i : 0 \le i < j : x^{2i} \rangle \quad .$$

Hence, the equality also holds in R. Also, the semiring R has 2^{n-1} distinct elements since each element in R has two representations as a polynomial in $GF(2)[x]$ with degree less than n, and there are 2^n such polynomials. (For example, $\mathbf{0}$ is represented by the two polynomials $\langle \Sigma i : 0 \le i < n : 0 \times x^i \rangle$ and $\langle \Sigma i : 0 \le i < n : 1 \times x^i \rangle$.) In particular (cf (4))

$$x^k =_R \langle \Sigma i : 0 \le i < n \wedge i \ne k : x^i \rangle \quad .$$

Consequently, if the function $\#$ of type $GF(2)[x] \to \mathbb{N}$ counts the number of non-zero coefficients in a given polynomial, then for all k and all P in $GF(2)[x]$ with degree less than n,

$$(8) \qquad (P =_R x^k) \;\Rightarrow\; (\#P = 1) \vee (\#P = n-1) \quad .$$

(It is at this point that the subscript R on the equality sign becomes essential; the left and right side of the equation denote P and x^k, respectively, after injection into the semiring $GF(2)[x]$ modulo the polynomial $\langle \Sigma i : 0 \le i < n : x^i \rangle$.)

We now have:

$$
\begin{array}{ll}
 & (4) \\
= & \{ \quad (6) \text{ and } (7) \quad \} \\
 & \langle \exists k :: \langle \Sigma i : 0 \le i < m \,\mathsf{mod}\, n : x^{2i} \rangle =_R x^k \rangle \quad .
\end{array}
$$

In order to apply (8), we conduct a case analysis on $m \,\mathsf{mod}\, n$ and on n. There are three cases to consider:

(a) $2(m \,\mathsf{mod}\, n) < n$.
This is the easiest case. The degree of $\langle \Sigma i : 0 \le i < m \,\mathsf{mod}\, n : x^{2i} \rangle$ is less than n so, applying (8), we have:

$$(4) \quad \Rightarrow \quad (m \,\mathsf{mod}\, n = 1) \vee (m \,\mathsf{mod}\, n = n{-}1) \quad .$$

(b) $2 \times (m \,\mathsf{mod}\, n) \ge n \wedge \mathsf{even}.n$.
Suppose $n = 2q$. The goal is to reduce $\langle \Sigma i : 0 \le i < m \,\mathsf{mod}\, n : x^{2i} \rangle$ to a polynomial with degree less than n. This is done in the following calculation. (Equalities are in R, i.e. in $GF(2)[x] \ / \ \langle \Sigma i : 0 \le i < n : x^i \rangle$.)

$$
\begin{array}{ll}
 & \langle \Sigma i : 0 \le i < m \,\mathsf{mod}\, n : x^{2i} \rangle \\
=_R & \{ \quad \text{assumption, } 2(m \,\mathsf{mod}\, n) \ge n \quad \} \\
 & \langle \Sigma i : 0 \le i < q : x^{2i} \rangle \ + \ \langle \Sigma i : q \le i < m \,\mathsf{mod}\, n : x^{2i} \rangle \\
=_R & \{ \quad (5),\ n = 2q \quad \} \\
 & \langle \Sigma i : 0 \le i < q : x^{2i} \rangle \ + \ \langle \Sigma i : q \le i < m \,\mathsf{mod}\, n : x^{2(i-q)} \rangle \\
=_R & \{ \quad \text{quantifier calculus, in } GF(2) \text{, } [\, a{+}a = 0 \,] \text{,} \\
 & \quad\ \ m \,\mathsf{mod}\, n < n = 2q \quad \} \\
 & \langle \Sigma i : m \,\mathsf{mod}\, n - q \le i < q : x^{2i} \rangle \quad .
\end{array}
$$

The last line above is a polynomial in $GF(2)$ with degree less than n. Hence, applying (8) to it, we have:

$$(4) \quad \Rightarrow \quad (2q - m \,\mathsf{mod}\, n = 1) \vee (2q - m \,\mathsf{mod}\, n = n{-}1) \quad .$$

Simplifying, using $n = 2q$ (and symmetry of disjunction),

$$(4) \quad \Rightarrow \quad (m \,\mathsf{mod}\, n = 1) \vee (m \,\mathsf{mod}\, n = n{-}1) \quad .$$

(c) $2 \times (m \,\mathsf{mod}\, n) \ge n \wedge \mathsf{odd}.n$.
Suppose $n = 2q - 1$. Then, by a similar calculation, we have:

$$
\begin{array}{ll}
 & \langle \Sigma i : 0 \le i < m \,\mathsf{mod}\, n : x^{2i} \rangle \\
=_R & \{ \quad \text{assumption, } 2(m \,\mathsf{mod}\, n) \ge n \text{,} \\
 & \quad\ \ (5),\ n = 2q - 1 \quad \} \\
 & \langle \Sigma i : 0 \le i < q : x^{2i} \rangle \ + \ \langle \Sigma i : q \le i < m \,\mathsf{mod}\, n : x^{2(i-q)+1} \rangle \\
=_R & \{ \quad \text{quantifier calculus} \quad \} \\
 & \langle \Sigma i : 0 \le i < m \,\mathsf{mod}\, n : x^{i} \rangle \quad .
\end{array}
$$

The last line above is a polynomial in $GF(2)$ with degree less than n. Hence, applying (8) to it, we again get:

$$(4) \quad \Rightarrow \quad (m \,\mathsf{mod}\, n = 1) \vee (m \,\mathsf{mod}\, n = n{-}1) \quad .$$

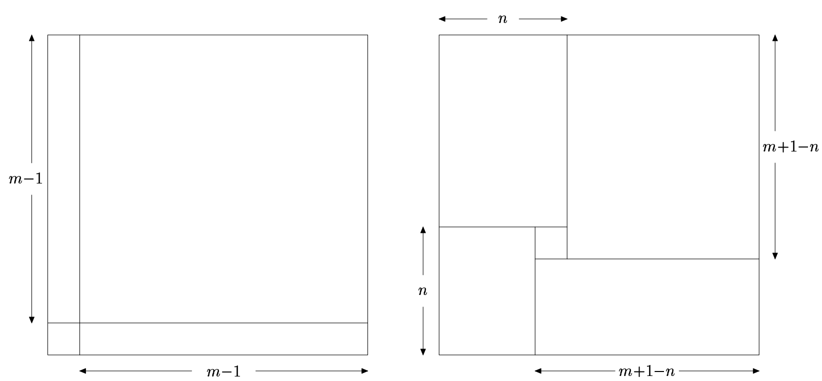
We conclude that, in all cases, (4) implies that

$$(m \,\mathsf{mod}\, n \ = \ 1) \ \vee \ (m \,\mathsf{mod}\, n \ = \ n{-}1) \quad .$$

Equivalently,

$$n\backslash(m{-}1) \ \vee \ n\backslash(m{+}1) \quad .$$

Figs. 2(a) and 2(b) show that this condition is also sufficient.



(a) $n\backslash(m{-}1)$

(b) $n\backslash(m{+}1)$

Fig. 2. $n\backslash(m{-}1) \vee n\backslash(m{+}1)$ is sufficient

4 Games On Cyclotomic Polynomials

In this section, we solve a novel class of games played on a one-dimensional tape. The general class is considered in section 4.2; the so-called "nuclear pennies game" [9] based on the "seven trees in one" property [10, 11] is used in section 4.1 to introduce the solution method.

4.1 Seven-Trees-in-One and the Nuclear Pennies Game

Consider the definition of binary trees — a binary tree is an empty tree or an element together with a pair of binary trees. Let us use symbols $+$ and $\times$ to denote disjoint union and Cartesian product respectively and let $\mathbb{1}$ denote the unit type. The type T of binary trees can be characterised by the type isomorphism $T \cong \mathbb{1}+T\times T$. Surprisingly, it can be shown that there is an isomorphism between seven-tuples of binary trees and binary trees. That is, $T^7 \cong T$. This has been dubbed "seven trees in one" by Blass [10] who attributes the isomorphism to a remark made by Lawvere [12].

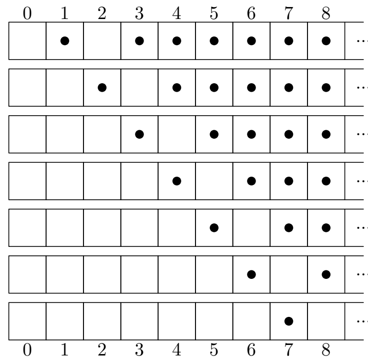
The isomorphism has been turned into a game with checkers called the "nuclear pennies game" [3]. The game is played on a one-dimensional board of infinite extent. A checker is placed on one of the squares and the goal is to move the checker six places to the right. An atomic move is to replace a checker in a square numbered $n+1$ by two checkers, one in each of the two adjacent squares n and $n+2$, or vice-versa, two checkers, one in square n and one in square $n+2$ for some n, are replaced by a checker in square $n+1$. The connection with seven-trees-in-one is easy to see if one views a move as replacing $T^n \times T$ by $T^n \times (\mathbb{1}+T\times T)$ or vice-versa.

The nuclear-pennies game has an easy solution if one exploits the left-right symmetry of the problem (moving a coin 6 places to the right is the same as moving a coin 6 places to the left). The problem is decomposed into first ensuring that there is a checker in the square 6 places to the right of the starting position and, symmetrically, there is a checker in the square 6 places to the left of the finishing position.

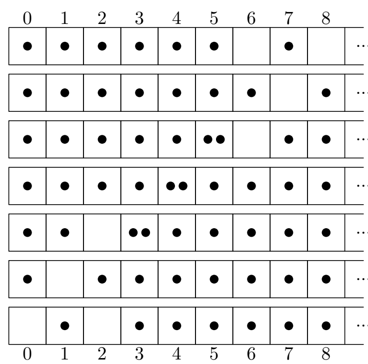
Achieving this first stage is easy. Below we show how it is done. First, six moves are needed to ensure that a checker is added six places to the right of the starting position. (This is shown below using dots to indicate checkers on a square. A blank indicates no checker on the square.)

0	1	2	3	4	5	6	7	8	
	•								...
•		•							...
•	•		•						...
•	•	•		•					...
•	•	•	•		•				...
•	•	•	•	•		•			...
•	•	•	•	•	•		•		...
0	1	2	3	4	5	6	7	8	

Symmetrically, working from bottom to top, six moves are needed to ensure that a checker is added six places to the left of the finishing position.



Now the goal is to connect these two intermediate states (the bottom state in the top diagram and the top state in the bottom diagram). An appropriate (symmetrical) sequence of states is as follows. (For the reader's convenience, the last and first states in the above figures are repeated as the top and bottom states in the figure below.)



The first and last moves make the number of checkers in the leftmost and rightmost positions equal. Then a small amount of creativity is needed to identify the two (symmetrical) moves to the (symmetrical) middle state.

4.2 Cyclotomic Polynomials

Although the nuclear-pennies game is an interesting exercise in the exploitation of symmetry in problem decomposition, it has until recently been an isolated example and appears to have attracted relatively little attention. (The website [9] gives one other example, dubbed the "thermonuclear pennies game".) In a recently submitted paper, Chen and Backhouse [4] posed the problem of, given an arbitrary n, is it possible to invent an "interesting" type T such that $T \cong T^{n+1}$. Likewise, given an arbitrary n, is it possible to invent an "interesting" nuclear-pennies-like game in which the task is to move a checker n places to the right using a sequence of pre-defined atomic moves. They gave an affirmative solution to the first question and a partial solution to the second, both answers being based on the use of so-called cyclotomic polynomials. (The solution to the second problem is partial in the sense that games were invented for an unbounded number of values of n but not all numbers n.) In this section, we present this class of “cyclotomic” games and their solution. (The paper [4] predicts that all cyclotomic games are solvable but does not give an explicit solution.)

The games we consider in this section are all based on an equation of the form

$$(9) \qquad T^1 = T^1 + \Gamma$$

where Γ is a polynomial in T with positive integer coefficients. The atomic moves in such a game are to supplement a checker at position $i+1$ by Γ_k additional checkers at positions $i+k$ where Γ_k is the coefficient of T^k in the polynomial Γ —this corresponds to the use of the equation (9) as a left-to-right replacement rule— or, vice-versa, if there is at least one checker at position $i+1$ and additionally Γ_k checkers at each position $i+k$, remove the Γ_k checkers at the positions $i+k$. The task is to move from an initial state in which there is just one checker at position 1 to a final state where there is just one checker at position n, for some pre-defined n.

A concrete example is the game based on the equation:

$$(10) \qquad T^1 \;=\; T^1 + (T^0 + T^4) \quad .$$

In this game, we are given the following board with one checker on position 1:

0	1	2	3	4	5	6	7	8	9	10	11	12	
	•												...

Now, whenever there is a checker in position $i+1$, we are allowed to add two checkers to the board —one in position i and the other in position $i+4$— and, vice-versa, whenever there is at least one checker in positions i, $i+1$, and $i+4$, we are allowed to remove one checker from positions i and $i+4$. The question is to determine if, from the initial state shown above, it is possible to move the checker 8 places to the right, i.e., to obtain the following state:

0	1	2	3	4	5	6	7	8	9	10	11	12	
									•				...

A necessary condition for a game based on (9) to be solvable is easily determined. Suppose we represent any state of the board by a polynomial in $\mathbb{Z}[T]$. Then, the moves are so designed that the polynomial modulo Γ is an invariant. The initial state is represented by T and the desired final state is represented by T^{n+1}. A necessary condition is thus that

$$(11) \qquad T \,\mathsf{mod}\, \Gamma \;=\; T^{n+1} \,\mathsf{mod}\, \Gamma \quad .$$

Equivalently, the polynomial $T^{n+1} - T$ must be divisible by Γ.

A well-known result is that, in $\mathbb{Z}[T]$, a polynomial is a divisor of $T^{n+1} - T$ equivales it is a product of so-called cyclotomic polynomials. (See Wikipedia or [4] for further information on cyclotomic polynomials.) For our purpose of inventing games with checkers, we restrict attention to products of cyclotomic polynomials in $\mathbb{N}[T]$. Specifically, we consider the polynomials $\psi_{a,n}$ where

$$\psi_{a,n} \;=\; \langle \Sigma i : 0 \le i < a : T^{i \times a^{n-1}} \rangle \qquad .$$

We assume that a and n are so chosen that the degree of $\psi_{a,n}$ is at least 2 . Equivalently, we assume that

$$(12) \qquad (2 \le a \wedge 2 \le n) \;\vee\; (3 \le a \wedge 1 = n) \;\; .$$

(When a is a prime number, $\psi_{a,n}$ is a cyclotomic polynomial and is commonly denoted by Φ_{a^n} . When a is not prime, $\psi_{a,n}$ is a product of cyclotomic polynomials.) For a game based on $\psi_{a,n}$ the goal is to move a checker a^n places to the right. For example, $\psi_{2,3} = 1{+}T^4$; it is this polynomial that is used in the game defined by (10) with goal to move the checker 2^3 (i.e. 8) places to the right.

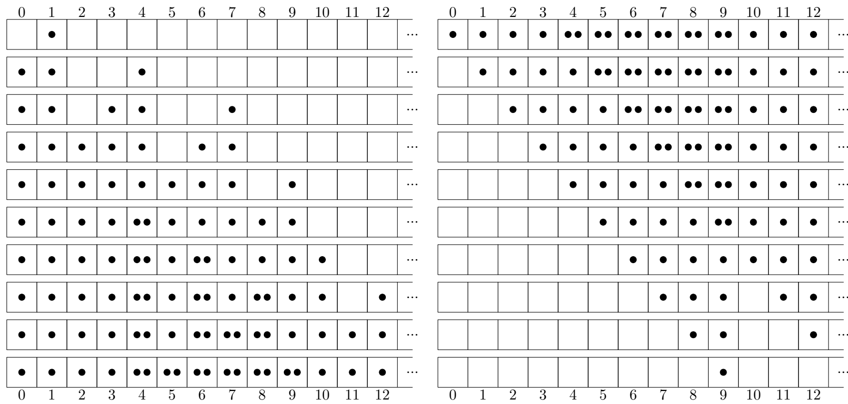
Before considering the general case, let us see how we could move the checker in the above example. The first trivial observation is that we have to add new checkers to the board (whenever we add checkers, we say that we perform an *expansion*). Another trivial observation is that, in order to leave a single checker in position 9 , we have to apply the rule in reverse, i.e., we have to remove checkers (whenever we remove checkers, we say that we perform a *contraction*). From these two observations, we propose constructing an algorithm that is divided into two phases: first, we perform a sequence of expansions that place at least one checker in the desired position; second, we perform a sequence of contractions until we have exactly one checker in the desired position. Figure 3 shows a solution. Note the division into two distinct phases: fig. 3(a) shows a sequence of expansions and fig. 3(b) shows a sequence of contractions. (For the reader's convenience, the middle state is repeated at the bottom of fig. 3(a) and the top of fig. 3(b).)

We now consider the general problem. As it turns out, our solution involves a case analysis on the values of a and n . There are two cases that exhibit the same sort of symmetry as the "nuclear pennies game" and can be solved using the same strategy as was used for that game. These are (a) $2{=}a \wedge 2{=}n$ and (b) $3{=}a \wedge 1{=}n$. In case (a), $\psi_{a,n} = \psi_{2,2} = 1{+}T^2$. In case (b), $\psi_{a,n} = \psi_{3,1} = 1{+}T{+}T^2$. We leave these cases as elementary exercises for the reader. (Exploit the symmetry in the polynomials about T^1 and the left-right symmetry of the task. See section 4.1 for the strategy.) We split the remaining cases into (c) $4 \le a \wedge 1 = n$ and (d) $2 \le a \wedge 2 \le n \wedge (2{\ne}a \vee 2{\ne}n)$.

Algorithm Decomposition Considering the discussion in the previous section and motivated by the example shown in figure 3, we specify the algorithm we want to develop as follows:

$\{ \quad s = T \quad \}$

perform a sequence of expansions



(a) First phase: sequence of expansions (positions 1, 4, 3, 6, 5, 7, 9, 8, and 6)

(b) Second phase: sequence of contractions (all positions from 1 upto 9)

Fig. 3. Moving a checker 8 places to the right when the move is given by $T = T + \psi_{2,3}$

$\{ \quad s =$ "some intermediate state" $\}$;

perform a sequence of contractions

$\{ \quad s = T^{a^n+1} \quad \}$.

Because moves are given by the equation $T = T + \psi_{a,n}$, we can model the *expansion* of position $k+1$ as

$$s \;:=\; s + T^k \times \psi_{a,n} \quad ,$$

and we can model the *contraction* of position $k+1$ as

$$s \;:=\; s - T^k \times \psi_{a,n} \quad .$$

This allows us to refine the specification:

$\{ \quad s = T \quad \}$

do choose appropriate k;

$\quad s \;:=\; s + T^k \times \psi_{a,n}$

od

$\{ \quad s =$ "some intermediate state" $\}$;

do choose appropriate k;

$\quad s \;:=\; s - T^k \times \psi_{a,n}$

od

$\{ \quad s = T^{a^n+1} \quad \}$.

The Intermediate State Let us explore the "intermediate state". An invariant of the first phase is that $s-T$ is divisible by $\psi_{a,n}$; an invariant of the second phase is that $s-T^{a^n+1}$ is divisible by $\psi_{a,n}$. So the "intermediate state" must be a polynomial s such that both $s-T$ and $s-T^{a^n+1}$ are divisible by $\psi_{a,n}$. Now,

$$
\begin{aligned}
& T^{a^n+1} - T \\
= \quad & \{ \quad \text{geometric series} \quad \} \\
& (T^{a^{n-1}+1} - T) \times \langle \Sigma i : 0 \le i < a : T^{i \times a^{n-1}} \rangle \\
= \quad & \{ \quad \text{definition of } \psi_{a,n} \quad \} \\
& (T^{a^{n-1}+1} - T) \times \psi_{a,n} \quad .
\end{aligned}
$$

It follows that, for all γ in $\mathbb{N}[T]$,

$$ T \ + \ (\gamma + T^{a^{n-1}+1}) \times \psi_{a,n} \quad = \quad T^{a^n+1} \ + \ (\gamma + T) \times \psi_{a,n} \quad . \tag{13} $$

The left and right sides of this equation express our "intermediate state": the left side is the postcondition of the expansion phase and the right side is the precondition of the contraction phase. Our task is to choose γ in such a way that the intermediate state can be reached both by a sequence of expansions and the reverse of a sequence of contractions.

Contraction Phase We start with the contraction phase which is much simpler. Note that 1 is a term in the polynomial $\psi_{a,n}$. This means that if an expansion is applied to position $k+1$ a checker is introduced at position k . Then an expansion can be applied to position k , introducing a checker at position $k-1$. And, of course, so on ad infinitum. In this way, a sequence of expansions starting from the state T^{a^n+1} (a checker in position a^n+1) and applied to positions a^n+1 , a^n , $\ldots$, 1 will yield the state

$$ T^{a^n+1} \ + \ \langle \Sigma i : 0 \le i < a^n+1 : T^i \rangle \times \psi_{a,n} $$

which has the form of the right side of (13). Reversing this process gives us the contraction phase:

$\{ \quad s \ = \ T^{a^n+1} + \langle \Sigma i : 0 \le i < a^n+1 : T^i \rangle \times \psi_{a,n} \quad \}$

$k := 0;$

$\{$ **Loop invariant:** $s = T^{a^n+1} + \langle \Sigma i : k \le i < a^n+1 : T^i \rangle \times \psi_{a,n} \quad \}$

do $\quad k < a^n+1 \ \rightarrow \ s\,,k \ := \ s - T^k \times \psi_{a,n} \ , \ k+1$

od

$\{ \quad s = T^{a^n+1} \quad \}$

Expansion Phase We now have to construct the loop corresponding to the expansion phase. The postcondition of the expansion phase is the precondition of the contraction phase. Because of (13), this is

$$s \;=\; T \;+\; (\langle \Sigma i \,:\, 0 \leq i < a^n{+}1 \wedge i \neq 1 \,:\, T^i\rangle \;+\; T^{a^{n-1}+1}) \times \psi_{a,n} \quad .$$

The precondition is $s = T$. Recalling the definition of an expansion of position $k{+}1$, we are required to expand each position in $\{1\} \cup \{i : 2 \leq i < a^n{+}1 : i{+}1\}$ once together with the position $a^{n-1}{+}2$ a second time. Of course, since the initial state is that there is one checker at position 1, the first move is to expand position 1. Let $\mathcal{E}$ denote the bag of positions remaining to be expanded. That is,

$$\mathcal{E} \;=\; \{i : 2 \leq i < a^n{+}1 : i{+}1\} \uplus \{a^{n-1}{+}2\} \quad .$$

(The symbol "$\uplus$" denotes bag union.) Then, the expansion phase after the first move has been completed is implemented as follows.

$\{ \quad s \;=\; T{+}\psi_{a,n} \quad \}$

$A\,,B \;:=\; \mathcal{E}\,,\emptyset\;;$

$\{$ **Loop invariant:** $\quad s = T + \langle \Sigma i : i \in \{1\} \uplus B : T^{i-1}\rangle \times \psi_{a,n}$

$\qquad \wedge \quad A \uplus B = \mathcal{E} \quad \}$

do $A \neq \emptyset \rightarrow$ choose j such that $j{\in}A$ and there is a checker in position j ;

$\qquad s\,,A\,,B \;:=\; s + T^{j-1} \times \psi_{a,n}\;,\; A{-}\{j\}\;,\; B \uplus \{j\}$

od

$\{ \quad s = T + (T^{a^{n-1}+1} + \langle \Sigma i \,:\, 0 \leq i < a^n{+}1 \wedge i \neq 1 \,:\, T^i\rangle) \times \psi_{a,n} \quad \}$

In order to expand a position it is required that there be a checker on that position. The correctness of the algorithm depends therefore on showing that, at each iteration, it is possible to choose a suitable value of j. (Formally, the "choose" statement is a conditional statement which will abort if j cannot be chosen.) Since the expansion phase never removes checkers (unlike the nuclear pennies game), it suffices to show that there is at least one way of ordering the elements of the bag $\mathcal{E}$ that guarantees that the position is occupied when the element is chosen. Such an ordering we called a *valid* ordering.

In the case that $1 = n$, the correctness of the algorithm is obvious. The bag $\mathcal{E}$ is then

$$\{i : 3 \leq i < a{+}2 : i\} \uplus \{3\} \quad .$$

Position 3 has to be expanded twice (since $2 \leq a$) and each of the positions 4, $\ldots$, $a{+}1$ have to be expanded once. Assuming that $4 \leq a$, there is indeed a checker at position 3 and the position can be expanded. This ensures that there are checkers at all positions 0, 1, $\ldots$, $a{+}1$. Subsequently, the positions 3, $\ldots$, $a{+}1$ can be expanded in an arbitrary order. (In other words, any ordering that places position 3 first is a valid ordering.)

The second case, when $2 \leq a \wedge 2 \leq n \wedge (2 \neq a \vee 2 \neq n)$, is the most difficult. The construction of a valid ordering reuses the central idea of the contraction phase, namely that once a checker has been introduced at position $k{+}1$ we can always expand positions k, $k{-}1$, $k{-}2$, etc.

In the following calculation, we determine a valid ordering for the expansions. Note the assumption $3 \leq a^{n-1}$ in the first step . This is why we perform a case analysis. (The assumption is indeed satisfied when $2 \leq a \wedge 2 \leq n \wedge (2 \neq a \vee 2 \neq n)$.)

$$
\begin{array}{ll}
 & T + T^0 \times \psi_{a,n} \\
= & \{ \quad \text{assuming } 3 \leq a^{n-1} \text{, the coefficient of } T^{a^{n-1}} \text{ in } T^0 \times \psi_{a,n} \text{ is } 1 \\
 & \quad\;\; \text{so we can expand positions } a^{n-1},\ a^{n-1}{-}1,\ \ldots,\ 4,\ 3 \quad \} \\
 & T \,+\, T^0 \times \psi_{a,n} \,+\, \langle \Sigma i : 2 \leq i < a^{n-1} : T^i \rangle \times \psi_{a,n} \\
= & \{ \quad \text{now the coefficient of } T^{a^{n-1}+2} \text{ in } T^2 \times \psi_{a,n} \text{ is } 1 \\
 & \quad\;\; \text{so we can expand positions } a^{n-1}{+}2 \text{ and } a^{n-1}{+}1 \quad \} \\
 & T \,+\, T^0 \times \psi_{a,n} \,+\, \langle \Sigma i : 2 \leq i < a^{n-1}{+}2 : T^i \rangle \times \psi_{a,n} \\
= & \{ \quad \langle \Sigma i : 2 \leq i < a^{n-1}{+}2 : T^i \rangle \times \psi_{a,n} \;=\; \langle \Sigma i : 0 \leq i < a^n : T^{i+2} \rangle \quad \} \\
 & T \,+\, T^0 \times \psi_{a,n} \,+\, \langle \Sigma i : 2 \leq i < a^n{+}2 : T^i \rangle \\
= & \{ \quad \text{now expand positions } a^{n-1}{+}2,\ a^{n-1}{+}3,\ \ldots,\ a^n{+}1 \quad \} \\
 & \quad T \,+\, T^0 \times \psi_{a,n} \,+\, \langle \Sigma i : 2 \leq i < a^n{+}2 : T^i \rangle \\
 & + \quad \langle \Sigma i : a^{n-1}{+}1 \leq i < a^n{+}1 : T^i \rangle \times \psi_{a,n} \\
= & \{ \quad \langle \Sigma i : 0 \leq i < a^n : T^{i+2} \rangle \;=\; \langle \Sigma i : 2 \leq i < a^{n-1}{+}2 : T^i \rangle \times \psi_{a,n} \quad \} \\
 & \quad T \,+\, T^0 \times \psi_{a,n} \,+\, \langle \Sigma i : 2 \leq i < a^{n-1}{+}2 : T^i \rangle \times \psi_{a,n} \\
 & + \quad \langle \Sigma i : a^{n-1}{+}1 \leq i < a^n{+}1 : T^i \rangle \times \psi_{a,n} \\
= & \{ \quad \text{range splitting} \quad \} \\
 & T + (\langle \Sigma i \;:\; 0 \leq i < a^n{+}1 \wedge i \neq 1 \;:\; T^i \rangle \,+\, T^{a^{n-1}+1}) \times \psi_{a,n} \quad .
\end{array}
$$

In summary, a valid ordering of expansions is, first, position 1, then positions a^{n-1}, $a^{n-1}{-}1$, $\ldots$, 4, 3, then positions $a^{n-1}{+}2$ and $a^{n-1}{+}1$ and finally positions $a^{n-1}{+}2$, $a^{n-1}{+}3$, $\ldots$, $a^n{+}1$.

This completes the expansion phase and the algorithm.

5 Conclusion

The games presented here were developed in order to support teaching of invariant properties in introductory, university-level courses. The current presentation of their solution is possibly too difficult for that level but they could be used to support more advanced algorithmically oriented courses that depend on polynomial arithmetic. (Coding theory is an obvious example.)

The common theme of all our examples is the exploitation of simple invariant properties of polynomials. It is this insight that enabled us to invent the cyclotomic games in section 4.2 (which we believe to be original to this paper). We are currently trying to find a complete characterisation of games based on the equation

$$T^k = T^k + \Gamma$$

where k is a positive number and Γ is a polynomial in T with positive integer coefficients. (The generalisation from 1 to k makes the contraction phase harder.)

Some improvement on our derivations would be welcome. We have been obliged to use case analyses in both the solution of the tiling problem (section 3.2) and the solution of the cyclotomic games (section 4.2). This is unfortunate but apparently unavoidable.

The class of tiling problems we have discussed assumes that all "ominoes" are straight. Golomb [1] extends his colouring argument to other problems where the ominoes are not straight. We haven't explored these problems. It would be interesting to see whether the algebraic formulation can be extended to such problems in a uniform way.

The interested reader may want to prove that our solutions to the cyclotomic games minimise the number of moves. The generalisation of the games to higher dimensions may also be of interest.

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