Which Mathematics for the Information Society?

(Project MathIS)

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2nd Int. FME Conference on Teaching Formal Methods (TFM2009)
Eindhoven, 6 November 2009

Introduction

Mathematics has the dubious honor of being the least popular subject in the curriculum. . . Future teachers pass through the elementary school learning to detest mathematics. . . They return to the elementary school to teach a new generation to detest it.

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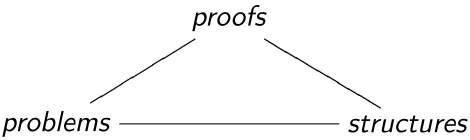
Dynamics of Mathematics

A perspective on the dynamics of mathematics



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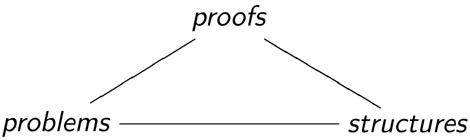

However...

- proofs are usually omitted from school manuals and teaching practice
- when present, proofs are non-systematic, informal, and seen as difficult

Contributions of Computing Science

Three important topics

- central role of formal logic
- development of a calculational style of reasoning
- emphasis on algorithmic content

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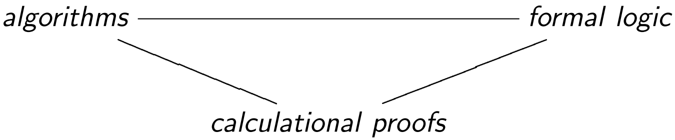

A central role for formal logic

We propose:

- earlier introduction and explicit use of logic
- introduction of logic via recreational problems that emphasize formalization and calculation
- introduction of simultaneous equations on booleans by analogy with simultaneous equations on numbers

Simultaneous equations

Suppose Ben is twice as old as Anne, but two years ago, Ben was three times as old as Anne. How old are Ben and Anne?

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$$(b = 2\times a) \;\; \wedge \;\; (b-2 = 3\times(a-2))$$

Portia's Casket

Portia had two caskets: gold and silver. Inside one of these caskets Portia had put her portrait and on each was an inscription. Portia explained to her suitor that each inscription could be either true or false but on the basis of the inscriptions he was to choose the casket containing the portrait. If he succeeded he could marry her. The inscriptions were:

Gold:	Exactly one of these inscriptions is true.
Silver:	This inscription is true if the portrait is in here.

Which casket contained the portrait? What can we deduce about the inscriptions?

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$$(g \equiv (g \equiv \neg s)) \;\wedge\; (s \equiv s \Leftarrow S) \;\wedge\; (G \equiv \neg S)$$

A calculational reasoning style

We propose:

- emphasis on systematic mathematical calculation
  - "Don't guess. Calculate!"
- systematic proof format

$$
\begin{array}{lll}
 & A & \\
= & & \{ \quad \text{hint why } A = B \quad \} \\
 & B & \\
\Leftarrow & & \{ \quad \text{hint why } B \Leftarrow C \quad \} \\
 & C &
\end{array}
$$

- replacement of *implication-first logic* by *equational logic*

A calculational reasoning style

From the book "Elementary Number Theory" by Gareth and Mary Jones (page 39):

Lemma 3.1
For any fixed $n \geq 1$ *we have* $a = b \pmod{n}$ *if and only if* $n \backslash (a - b)$.

Proof (by mutual implication)
Putting $a = q \times n + r$ *and* $b = q' \times n + r'$ *as above, we have* $a - b = (q - q') \times n + (r - r')$ *with* $-n < r - r' < n$*. If* $a = b \pmod{n}$ *then* $r - r' = 0$ *and* $a - b = (q - q') \times n$*, which is divisible by* n*. Conversely, if* n *divides* $a - b$ *then it divides* $(a - b) - (q - q') \times n = r - r'$*; now the only integer strictly between* $-n$ *and* n *which is divisible by* n *is* 0*, so* $r - r' = 0$*, giving* $r = r'$ *and hence* $a = b \pmod{n}$*.*

A calculational reasoning style

Writing $a = q \times n + r$ *and* $b = q' \times n + r'$, *we have* $a - b = (q - q') \times n + (r - r')$, *with* $-n < r - r' < n$. *Hence,*

$$
\begin{array}{ll}
 & n \backslash (a - b) \\
= & \{ \quad \textit{consideration above} \quad \} \\
 & n \backslash ((q - q') \times n + (r - r')) \\
= & \{ \quad \textit{division property} \quad \} \\
 & n \backslash (r - r') \\
= & \{ \quad -n < r - r' < n \quad \} \\
 & r - r' = 0 \quad .
\end{array}
$$

Therefore $r = r'$ *and, by definition,* $a = b \ (\mathrm{mod}\ \ n)$.

Making the algorithmic contents explicit

Teaching algorithmic skills

- many mathematical problems are of algorithmic nature
- algorithmic techniques can have a tremendous impact
  - invariants, contracts, problem-decomposition, program inversion, program termination, etc.
- *computational thinking*: a PROBE on mathematics education

Making the algorithmic contents explicit

Euclid's algorithm computes $m \nabla n$

$\{\ 0 < m \ \wedge \ 0 < n \ \}$

$x\,,y \ := \ m\,,n \ ;$

$\{$ **Invariant**: $\quad m \nabla n = x \nabla y$

Bound function: $x + y \ \}$

$do \ \ x > y \ \rightarrow \ x := x - y$

$\Box \ \ y > x \ \rightarrow \ y := y - x$

od

$\{\ x = y \ \wedge \ m \nabla n = x = y \ \}$

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Prove: $fib.(n{+}1)\nabla fib.n = 1$

Making the algorithmic contents explicit

Proof that $fib.(n+1) \nabla fib.n = 1$

$x\,,y \;:=\; fib.(n+1)\,,fib.n\;;$

$\{$ **Invariant**: x and y are two consecutive fibonacci numbers $\wedge$

$fib.(n+1) \nabla fib.n = x \nabla y$

Bound function: $x + y \;\}$

$do\;\; x > y \;\;\rightarrow\;\; x := x - y$

$\square\;\; y > x \;\;\rightarrow\;\; y := y - x$

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$\{$ x and y are two consecutive fibonacci numbers $\wedge\; (x = y) \;\wedge$

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Making the algorithmic contents explicit

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$x , y \; := \; fib.(n+1) , fib.n \;;$

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An educational programme

We are currently planning and implementing a *pilot* educational programme:

- development of educational material on algorithmic problem solving
- teams of up to 20 volunteer secondary-school students, 15–17 years-old
- extra-curricular “Maths’ Clubs”, run by volunteer secondary-school teachers (sometimes by us)

Teaching Scenarios

Teaching scenarios are fully worked out solutions to problems together with "method sheets".

Structure of a teaching scenario

- brief description and goals
- problem statement
- students should know
- resolution
- **notes for the teacher**
- extensions and exercises
- further reading

Teaching Scenarios

Teaching scenarios are designed to promote **self-discovery**

- include obtrusive questions that the teacher should **not** ask
- include extensions to the problem and exercises
- include suggestions for further reading

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Example of an obtrusive question

Can we use Theorem X to solve the problem?

(when Theorem X is part of the solution)

Conclusions

Our view

- need to act at lower levels of the educational system
- computational thinking may have a decisive impact on reinvigorating mathematics education

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Educational material

- we have some educational material, including some novel results in number theory
- however, we need to create a lot more, if we want to succeed!

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- need to act at lower levels of the educational system
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Educational material

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- however, we need to create a lot more, if we want to succeed!

Tool support

- we are developing a structure editor for handwritten mathematics

Future Work

Next steps

- create more educational material
  - teaching scenarios, guides, books, etc.
- start with the math clubs

Assessment

- it will be difficult to assess the impact of our experiences
- method of assessment still not decided

Thank you!

http://www.di.uminho.pt/mathis